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Free vibration analysis of a rotating beam with nonlinear spring and mass system

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Abstract

Free, out of plane vibration of a rotating beam with nonlinear spring–mass system has been investigated. The nonlinear constraint is connected to the beam between two points on the beam through a rigid rod. Formulation of the equation of motion is obtained starting from transverse/axial coupling through axial strain. Solution is obtained by applying method of multiple time scale directly to the nonlinear partial differential equations and the boundary conditions. The results of the linear frequencies match well with those obtained in open literature. Subsequent nonlinear study indicates that there is a pronounced effect of spring and its mass. The influence of rigid rod location on frequencies is also investigated on nonlinear frequencies of rotating beam.

1. Introduction

Dynamic characteristics of rotating beam play a significant role in the overall performance and the design of various engineering systems, such as, turbo-machinery, wind turbines, robotic manipulators and rotorcraft blades. The determination of mode shape and natural frequencies of such rotating structures have been a topic of primary importance and as such received considerable attention from various researchers working in the related fields. Helicopter rotor blades are long, slender beams undergoing moderate deformation. During operation these blades experience large bending and centrifugal loads. There has been a continued effort to develop a mechanically simple yet efficient rotor blade and hub configuration. With the advancement in technology, the design and construction of these helicopter rotors has become very simple with the introduction of specialized elastomer with high loss factor [1,2], thus replacing the external hydraulic damper in the blade. Huber [3] presented a comprehensive review of the development of modern helicopter rotors in which elastomer plays an important role. The mechanical arrangement of the elastomeric damper leads to additional nonlinear constraint during the deformation of the blade. Consequently, the dynamic analysis of

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advanced helicopter rotors (known as bearingless rotor) becomes complicated due to multiple load paths and highly nonlinear characteristics of the elastomeric damper [4]. A study conducted by Hebert [5] utilized a set of distributed, tuned vibration absorbers to introduce lag damping in the rotor system. It has been inferred that this technique can be implemented on both articulated as well as bearingless rotors. Recently, A new model [6], based on Anelastic Displacement Fields, has been developed for the Elastomeric materials used in bearingless rotor to capture the frequency and amplitude-dependent behavior often exhibited by these materials.

Beam theories for moderate deformation have been developed by several researchers [7–10]. Following their approaches, one arrives at nonlinear analytical models which are ultimately used to obtain the equilibrium positions and subsequent linearized solutions.

Independent of the above studies related to helicopter blades, several researchers have made significant contribution to the study of nonlinear dynamics of beams using perturbation techniques. Anderson [11] formulated the nonlinear equation of motion of a rotating bar and obtained the natural frequencies from the linearized equation. Using a harmonic balance technique, the nonlinear structural dynamic analysis of blade model was performed by Minguet and Dugundji [12]. Nayfeh and his associates reported several studies to determine nonlinear response of stationary beams under large deflection. Nayfeh et al. [13] proposed a numerical perturbation method for the determination of nonlinear response of a continuous beam having complicated boundary conditions. The nonlinear response of a simple supported beam with an attached spring–mass system was also investigated by Pakdemirli and Nayfeh [14]. Nayfeh and Nayfeh [15] obtained the nonlinear modes and frequencies of a simply supported Euler–Bernoulli beam resting on an elastic foundation having quadratic and cubic nonlinearity. Recently, nonlinear normal mode shapes were determined for a cantilever beam by using the method of multiple time scale [16]. Pohit et al. [17–19] has modeled the characteristic of an elastomeric material and investigated the effect of nonlinear elastomeric constraint on rotating blade. They have applied a numerical perturbation technique to determine the frequency–amplitude relationship of a rotating beam under transverse vibration. Recently Pesheck et al. [20] proposed a method for determining reduced order models for rotating uniform cantilever Euler–Bernoulli beams.

Most of the studies on helicopter blades have focused primarily on the linearised dynamic analysis. Very little information is available on the influence of the elastomer on the structural dynamic characteristics of a rotor blade when the elastomer is included as a subsystem. Dowell [21] used the component mode analysis to examine the effect of material nonlinearity in the form of a nonlinear spring–mass system attached to a simply supported beam. Subsequently, Nayfeh and Nayfeh [15] and Pohit et al. [17], made attempts to study the effect of a nonlinear constraint on a simply supported and rotating beam respectively.

In a recent paper [22], formulation of equation of motion of a rotating beam with nonlinear constraint has been presented starting from transverse/axial coupling through axial strain. The nonlinear constraint with its mass appears in the boundary condition, thus making it possible to study the influence of spring–mass on the dynamic characteristic of the system.

The major objectives of the present paper are as follows:

(i) Formulation of equation of motion of a rotating beam with nonlinear constraint starting from transverse/axial coupling through axial strain. However, following the concept of bearingless rotor, the nonlinear constraint is connected to the beam between two points on the beam through a rigid rod.
(ii) Determination of nonlinear solution by applying methods of multiple time scale directly to the partial differential equations and the boundary conditions.
(iii) Study the influence of the location of the nonlinear constraint and its mass on nonlinear frequencies.

2. Formulation

The dynamics of rotating beam differs from that of non-rotating one due to the addition of centrifugal stiffness. The differential equations of motion for a rotating beam have variable co-efficient while those for a non-rotating beam have constant co-efficient. Additionally, in the present problem, there is a transverse constraint at the point B (Fig. 1) in the form of a nonlinear spring of mass M. The other end of the spring is attached to a rigid massless link EC, which is also rotating along with the beam AC. One end of the link being free and the other end is attached to the beam at C. Thus, Fig. 1 represents a simplified model of an otherwise very complicated bearingless rotor blade, in which elastomer occupies the position between points F and B. Since damping does not play any significant role as far as the natural frequencies are concerned, the elastomer connection is represented only by a spring element. It is to be noted that the deformation of the spring–mass system depends not only on the deflection of the beam at point B, but also on the deflection and slope at the point C. The motion is restricted to the transverse direction only, thereby eliminating lead lag and torsional motion, and allows axial strain. The effect of rotary inertia is also neglected.

Introducing $(\,)' \equiv$ space derivative with respect to x and $(\cdot) \equiv$ time derivative.

The expressions for the kinetic energy and the potential energy of the rotating beam are given in Eqs. (1) and (2), respectively.

$$
\begin{aligned}
\mathrm{KE} = T(t) = {} & \frac{1}{2}\int_0^{L_1} m\left[\left\{\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial w_1}{\partial t}\right)^2\right\} + \Omega^2(x+u_1)^2\right]\mathrm{d}x \\
& + \frac{1}{2}\int_{L_1}^{L_2} m\left[\left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial w_2}{\partial t}\right)^2 + \Omega^2(x+u_2)^2\right]\mathrm{d}x \\
& + \frac{1}{2}\int_{L_1}^{L_2} m\left[\left(\frac{\partial u_3}{\partial t}\right)^2 + \left(\frac{\partial w_3}{\partial t}\right)^2 + \Omega^2(x+u_3)^2\right]\mathrm{d}x \\
& + \left[\frac{1}{2}M\left\{\dot{w}_2(L_2) - (L_2 - L_1)\frac{\partial \dot{w}_2(L_2)}{\partial x} - \dot{w}_1(L_1)\right\}^2\right] \\
& + \left[\frac{1}{2}M\Omega^2\{x_s + u_1(L_1,t)\}^2\right],
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\mathrm{PE} = U(t) = {} & \frac{1}{2}\int_0^{L_1} EI(w_1'')^2\,\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2} EI(w_2'')^2\,\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L} EI(w_3'')^2\,\mathrm{d}x \\
& + \frac{1}{2}\int_0^{L_1} EA\left(u_1' + \frac{1}{2}w_1'^2\right)^2\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2} EA\left(u_2' + \frac{1}{2}w_2'^2\right)^2\mathrm{d}x \\
& + \frac{1}{2}\int_{L_1}^{L} EA\left(u_3' + \frac{1}{2}w_3'^2\right)^2 + \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\gamma\Delta^4,
\end{aligned} \tag{2}
$$

where extension of the spring Δ is given by

$$
\Delta = \left[w_2(L_2,t) - (L_2 - L_1)\frac{\partial w_2(L_2,t)}{\partial x} - w_1(L_1,t)\right].
$$

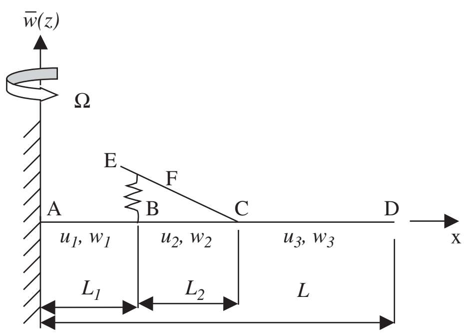

Fig. 1. Rotating beam with spring–mass system.

Here t denotes time, m denotes mass per unit length of beam, EI the flexural rigidity of the beam cross-section, Ω the angular velocity, A is the cross-sectional area of the beam, $w_i (i = 1, 2, 3)$ the transverse deflection at the three segments of the beam AD, $u_i (i = 1, 2, 3)$ are the left, central and right axial beam displacements, α and γ are the coefficients of the linear and nonlinear terms of the spring, respectively, M is the mass of the nonlinear spring and L is the length of the beam. The Eqs. (1) and (2) are now introduced into variational principle

$$\delta \int_{t_1}^{t_2} (T - U)\, \mathrm{d}t = 0. \tag{3}$$

Thus, one can obtain the governing equations and boundary conditions as follows:

$$-m \frac{\partial^2 u_1}{\partial t^2} + m\Omega^2 (x + u_1) + EA\left(u_1' + \frac{1}{2} w_1'^2\right)' = 0, \quad 0 \leqslant x \leqslant L_1, \tag{4a}$$

$$-m \frac{\partial^2 w_1}{\partial t^2} - EI w_1'''' + EA\left\{\left(u_1' + \frac{1}{2} w_1'^2\right) w_1'\right\}' = 0, \tag{4b}$$

$$-m \frac{\partial^2 u_2}{\partial t^2} + m\Omega^2 (x + u_2) + EA\left(u_2' + \frac{1}{2} w_2'^2\right)' = 0, \quad L_1 \leqslant x \leqslant L_2, \tag{5a}$$

$$-m \frac{\partial^2 w_2}{\partial t^2} - EI w_2'''' + EA\left\{\left(u_2' + \frac{1}{2} w_2'^2\right) w_2'\right\}' = 0 \tag{5b}$$

and

$$-m \frac{\partial^2 u_3}{\partial t^2} + m\Omega^2 (x + u_3) + EA\left(u_3' + \frac{1}{2} w_3'^2\right)' = 0, \quad L_2 \leqslant x \leqslant L, \tag{6a}$$

$$-m \frac{\partial^2 w_3}{\partial t^2} - EI w_3'''' + EA\left\{\left(u_3' + \frac{1}{2} w_3'^2\right) w_3'\right\}' = 0, \tag{6b}$$

$$\begin{aligned}
& w_1'(0,t) = 0, \quad w_1'(0,t) = 0, \quad w_3''(L,t) = 0, \quad w_3'''(L,t) = 0, \\
& EI w_1'''(L_1,t) - EI w_2'''(L_1,t) - EA\left(u_1' + \frac{1}{2} w_1'^2\right) w_1' + EA\left(u_2' + \frac{1}{2} w_2'^2\right) w_2' + \alpha\Delta + \gamma\Delta^3 + M\ddot{\Delta} = 0, \\
& w_1(L_1,t) = w_2(L_1,t), \quad w_1'(L_1,t) = w_2'(L_1,t), \quad w_1''(L_1,t) = w_2''(L_1,t), \\
& w_2(L_2,t) = w_3(L_2,t), \quad w_2'(L_2,t) = w_3'(L_2,t), \\
& - EI w_2'' + EI w_3'' + \alpha(L_2 - L_1)\Delta + \gamma(L_2 - L_1)\Delta^3 + (L_2 - L_1) M\ddot{\Delta} = 0 \quad \text{at } x = L_2, \\
& EI w_2''' - EI w_2''' - \alpha\Delta - \gamma\Delta^3 - M\ddot{\Delta} = 0 \quad \text{at } x = L_2, \\
& - EA\left(u_1' + \frac{1}{2} w_1'^2\right) + EA\left(u_2' + \frac{1}{2} w_2'^2\right) + M\Omega^2 (L_1 + u_1) = 0 \quad \text{at } x = L_1,
\end{aligned} \tag{7}$$

$$u_1(0,t) = 0, \quad u_1(L_1,t) = u_2(L_1,t), \quad u_2(L_2,t) = u_3(L_2,t), \quad u_2'(L_2,t) = u_3'(L_2,t), \quad u_3' + \frac{1}{2} w_3'^2 = 0 \quad \text{at } x = L. \tag{8}$$

The first four equations of (7) represent the end boundary conditions in the transverse mode of vibration of the rotating beam. The equations fifth to 12th of Eq. (7) describe the balance of shear forces, slopes and deflections in the transverse direction at $x = L_1$ and L_2, respectively. The 13th equation and the last equation of Eq. (8) indicate that axial deflections at $x = L_1$ and L are zero.

In the above derivation, the following assumptions are made

$$\frac{\partial w_1(0,t)}{\partial x}=0,\quad u_1(0,t)=0 \text{ or a constant},$$

$$\delta(w_1')=0 \quad \text{at } x=0 \quad \text{and} \quad w_1(0,t)=0.$$

At this stage, the following non-dimensional quantities are introduced.

$$x^*=\frac{x}{L},\quad w_i^*=\frac{w_i}{r},\quad u_i^*=\frac{u_i}{r},\quad \eta_1=\frac{L_1}{L},\quad \eta_2=\frac{L_2}{L},$$
$$a=\frac{EAr^2}{m\Omega^2L^4}=\frac{EI}{m\Omega^2L^4},\quad \varepsilon=\frac{r}{L},\quad t^*=\Omega t,\qquad i=1,2,3,$$

where r is the radius of gyration of the cross-section of the beam having length equal to L.

Governing equations and boundary conditions can be written as

$$\varepsilon^2\left(-\frac{\partial^2 u_1}{\partial t^2}+u_1\right)+x+a\left(u_1'+\frac{1}{2}w_1'^2\right)'=0,\quad 0\leqslant x\leqslant \eta_1, \tag{9a}$$

$$-\frac{\partial^2 w_1}{\partial t^2}-aw_1''''+a\left\{\left(u_1'+\frac{1}{2}w_1'^2\right)w_1'\right\}'=0, \tag{9b}$$

$$\varepsilon^2\left(-\frac{\partial^2 u_2}{\partial t^2}+u_2\right)+x+a\left(u_2'+\frac{1}{2}w_2'^2\right)'=0,\quad \eta_1\leqslant x\leqslant \eta_2, \tag{10a}$$

$$-\frac{\partial^2 w_2}{\partial t^2}-aw_2''''+a\left\{\left(u_2'+\frac{1}{2}w_2'^2\right)w_2'\right\}'=0 \tag{10b}$$

$$\varepsilon^2\left(-\frac{\partial^2 u_3}{\partial t^2}+u_3\right)+x+a\left(u_3'+\frac{1}{2}w_3'^2\right)'=0,\quad \eta_2\leqslant x\leqslant 1, \tag{11a}$$

$$-\frac{\partial^2 w_3}{\partial t^2}-aw_3''''+a\left\{\left(u_3'+\frac{1}{2}w_3'^2\right)w_3'\right\}'=0, \tag{11b}$$

$$-\left(u_1'+\frac{1}{2}w_1'^2\right)+\left(u_2'+\frac{1}{2}w_2'^2\right)+\alpha_3(\eta_1+\varepsilon^2u_1)=0 \quad \text{at } x=\eta_1,$$

$$u_1(0,t)=0,\quad u_1(\eta_1,t)=u_2(\eta_1,t),\quad u_2(\eta_2,t)=u_3(\eta_2,t),\quad u_2'(\eta_2,t)=u_3'(\eta_2,t),\quad u_3'+\frac{1}{2}w_3'^2=0 \quad \text{at } x=1,$$

$$w_1(0,t)=0,\quad w_1'(0,t)=0,\quad w_1(\eta_1,t)=w_2(\eta_1,t),\quad w_1'(\eta_1,t)=w_2'(\eta_1,t),$$

$$w_1'''(\eta_1,t)-w_2'''(\eta_1,t)-\left(u_1'+\frac{1}{2}w_1'^2\right)w_1'+\left(u_2'+\frac{1}{2}w_2'^2\right)w_2'+\alpha_1\Delta_1(\eta_1,t)+\alpha_2\Delta_1^3(\eta_1,t)+\alpha_3\ddot{\Delta}_1=0,$$

$$w_1''(\eta_1,t)=w_2''(\eta_1,t)=0,\quad w_2(\eta_2,t)=w_3(\eta_2,t),\quad w_2'(\eta_2,t)=w_3'(\eta_2,t),$$

$$-w_2''-w_3''+(\eta_2-\eta_1)\alpha_1\Delta_1+(\eta_2-\eta_1)\alpha_2\Delta_1^3+(\eta_2-\eta_1)\alpha_3\ddot{\Delta}_1=0 \quad \text{at } x=\eta_2,$$

$$w_2'''-w_3'''-\alpha_1\Delta_1-\alpha_2\Delta_1^3-\alpha_3\ddot{w}_1=0 \quad \text{at } x=\eta_2,$$

$$w_3''(1,t)=0,\quad w_3'''(1,t)=0. \tag{12}$$

It is to be noted that stars (*) are removed from all the quantities for convenience.

In the above equations, co-efficients α_1, α_2, α_3, Δ_1 and $\ddot{\Delta}_1$ are defined as follows:

$$\alpha_1 = \frac{\alpha L^3}{EI}, \quad \alpha_2 = \frac{\gamma r^2 L^3}{EI}, \quad \alpha_3 = \frac{M\Omega^2 L^3}{EI}, \quad \Delta_1 = \frac{\Delta}{r}, \quad \ddot{\Delta}_1 = \frac{\ddot{\Delta}}{r\Omega^2}.$$

Neglecting small order terms $\{O(\varepsilon^2)\}$, the governing equations may be written as

$$x + a\left(u_1' + \frac{1}{2}w_1'^2\right)' = 0, \quad 0 \leqslant x \leqslant \eta_1, \tag{13a}$$

$$-\frac{\partial^2 w_1}{\partial t^2} - aw_1'''' + a\left\{\left(u_1' + \frac{1}{2}w_1'^2\right)w_1'\right\}' = 0, \tag{13b}$$

$$x + a\left(u_2' + \frac{1}{2}w_2'^2\right)' = 0, \quad \eta_1 \leqslant x \leqslant \eta_2, \tag{14a}$$

$$-\frac{\partial^2 w_1}{\partial t^2} - aw_2'''' + a\left\{\left(u_2' + \frac{1}{2}w_2'^2\right)w_2'\right\}' = 0, \tag{14b}$$

$$x + a\left(u_3' + \frac{1}{2}w_3'^2\right)' = 0, \quad \eta_2 \leqslant x \leqslant 1, \tag{15a}$$

$$-\frac{\partial^2 w_3}{\partial t^2} - aw_3'''' + a\left\{\left(u_3' + \frac{1}{2}w_3'^2\right)w_3'\right\}' = 0, \tag{15b}$$

$$-\left(u_1' + \frac{1}{2}w_1'^2\right) + \left(u_2' + \frac{1}{2}w_2'^2\right) + \alpha_3\eta_1 = 0 \quad \text{at } x = \eta_1,$$

$$u_1(0,t) = 0, \quad u_1(\eta_1,t) = u_2(\eta_1,t), \quad u_2(\eta_2,t) = u_3(\eta_2,t), \quad u_2'(\eta_2,t) = u_3'(\eta_2,t), \quad u_3' + \frac{1}{2}w_3'^2 = 0 \quad \text{at } x = 1,$$

$$w_1(0,t) = 0, \quad w_1'(0,t) = 0, \quad w_1(\eta_1,t) = w_2(\eta_1,t), \quad w_1'(\eta_1,t) = w_2'(\eta_1,t),$$

$$w_1'''(\eta_1,t) - w_2'''(\eta_1,t) - \left(u_1' + \tfrac{1}{2}w_1'^2\right)w_1' + \left(u_2' + \tfrac{1}{2}w_2'^2\right)w_2' + \alpha_1\Delta_1(\eta_1,t) + \alpha_2\Delta_1^3(\eta_1,t) + \alpha_3\ddot{\Delta}_1 = 0,$$

$$w_1''(\eta_1,t) = w_2''(\eta_1,t), \quad w_2(\eta_2,t) = w_3(\eta_2,t), \quad w_2'(\eta_2,t) = w_3'(\eta_2,t),$$

$$-w_2'' - w_3'' + (\eta_2 - \eta_1)\alpha_1\Delta_1 + (\eta_2 - \eta_1)\alpha_2\Delta_2^3 + (\eta_2 - \eta_1)\alpha_3\ddot{\Delta}_1 = 0 \quad \text{at } x = \eta_2,$$

$$w_2''' - w_3''' - \alpha_1\Delta_1 - \alpha_2\Delta_1^3 - \alpha_3\ddot{w}_1 = 0 \quad \text{at } x = \eta_2,$$

$$w_3''(1,t) = 0, \quad w_3'''(1,t) = 0. \tag{16}$$

Integrating Eqs. (13a), (14a) and (15a), one gets

$$\frac{1}{2}x^2 + a\left(u_1' + \frac{1}{2}w_1'^2\right) = f_1(t), \quad 0 \leqslant x \leqslant \eta_1, \tag{17a}$$

$$\frac{1}{2}x^2 + a\left(u_2' + \frac{1}{2}w_2'^2\right) = f_2(t), \quad \eta_1 \leqslant x \leqslant \eta_2, \tag{17b}$$

$$\frac{1}{2}x^2 + a\left(u_3' + \frac{1}{2}w_3'^2\right) = f_3(t), \quad \eta_2 \leqslant x \leqslant 1. \tag{17c}$$

At $x = 0$, one obtains from Eq. (17a),

$$0 + au_1'(0,t) = f_1(t).$$

At $x = \eta_1$, one can write from Eqs. (17a) and (17b),

$$\frac{1}{2}\eta_1^2 + a\left(u_1' + \frac{1}{2}w_1'^2\right) - \frac{1}{2}\eta_1^2 - a\left(u_2' + \frac{1}{2}w_2'^2\right) = f_1(t) - f_2(t).$$

However, using the first boundary conditions of Eq. (16), one may write

$$a\alpha_3\eta_1 = f_1(t) - f_2(t).$$

One obtains from Eqs. (17b) and (17c)

$$\frac{1}{2}\eta_2^2 + a\left(u_2' + \frac{1}{2}w_2'^2\right) - \frac{1}{2}\eta_2^2 - a\left(u_3' + \frac{1}{2}w_3'^2\right) = f_2(t) - f_3(t).$$

However at $x = \eta_2$, $u_2' = u_3'$ and $w_2' = w_3'$
Therefore, $f_2(t) = f_3(t)$
At $x = 1$, one obtains from Eq. (17c),

$$\frac{1}{2} + a \times 0 = f_3(t) \Rightarrow f_3(t) = \frac{1}{2}.$$

Thus

$$f_2(t) = f_3(t) = \frac{1}{2} \quad \text{and} \quad f_1(t) = a\alpha_3\eta_1 + \frac{1}{2}. \tag{18}$$

Eliminating u_1' and u_2' from Eqs. (13)–(15), and with the help of Eq. (18), one gets the governing equations in transverse mode as

$$-\frac{\partial^2 w_1}{\partial t^2} - aw_1'''' - xw_1' + (a\alpha_3\eta_1)w_1'' + \frac{1}{2}(1-x^2)w_1'' = 0, \quad 0 \leqslant x \leqslant \eta_1,$$

$$-\frac{\partial^2 w_2}{\partial t^2} - aw_2'''' - xw_2' + \frac{1}{2}(1-x^2)w_2'' = 0, \quad \eta_1 \leqslant x \leqslant \eta_2,$$

$$-\frac{\partial^2 w_3}{\partial t^2} - aw_3'''' - xw_3' + \frac{1}{2}(1-x^2)w_3'' = 0, \quad \eta_2 \leqslant x \leqslant 1. \tag{19}$$

The corresponding boundary conditions are given as

$$w_1(0,t) = 0, \quad w_1'(0,t) = 0, \quad w_1(\eta_1,t) = w_2(\eta_1,t), \quad w_1'(\eta_1,t) = w_2'(\eta_1,t),$$

$$w_1'''(\eta_1,t) - w_2'''(\eta_1,t) - \alpha_3\eta_1 w_1' + \alpha_1\Delta_1 + \alpha_2\Delta_1^3 + \alpha_3\ddot{\Delta}_1 = 0 \quad \text{at } x = \eta_1,$$

$$w_1''(\eta_1,t) = w_2''(\eta_1,t), \quad w_2(\eta_2,t) = w_3(\eta_2,t), \quad w_2'(\eta_2,t) = w_3'(\eta_2,t),$$

$$-w_2'' - w_3'' + (\eta_2 - \eta_1)\alpha_1\Delta_1 + (\eta_2 - \eta_1)\alpha_2\Delta_2^3 + (\eta_2 - \eta_1)\alpha_3\ddot{\Delta}_1 = 0 \quad \text{at } x = \eta_2,$$

$$w_2''' - w_3''' - \alpha_1\Delta_1 - \alpha_2\Delta_1^3 - \alpha_3\ddot{w}_1 = 0 \quad \text{at } x = \eta_2,$$

$$w_3''(1,t) = 0, \quad w_3'''(1,t) = 0. \tag{20}$$

2.1. Solution methodology

For the subsequent solution of the above nonlinear equations, one may uses the method of multiple scale and seeks expansions of the solution for frequency-amplitude in the form

$$\begin{aligned} w_1(x,t;\varepsilon) &= \varepsilon w_{11}(x,T_0,T_2) + \varepsilon^3 w_{13}(x,T_0,T_2) + \varepsilon^5 w_{15}(x,T_0,T_2) + \cdots, \\ w_2(x,t;\varepsilon) &= \varepsilon w_{21}(x,T_0,T_2) + \varepsilon^3 w_{23}(x,T_0,T_2) + \varepsilon^5 w_{25}(x,T_0,T_2) + \cdots, \\ w_3(x,t;\varepsilon) &= \varepsilon w_{31}(x,T_0,T_2) + \varepsilon^3 w_{32}(x,T_0,T_2) + \varepsilon^5 w_{35}(x,T_0,T_2) + \cdots, \end{aligned} \tag{21}$$

where, w_{1n}, w_{2x} and w_{3n} are $O(1)$; ε is a small dimensionless parameter (defined earlier); $T_0 = t$ is a first time scale characterizing changes occurring at ω_n, where ω_n are the natural frequencies of the beam–spring system; and $T_2 = \varepsilon^2 t$ is a slow time scale, characterizing the modulation of the amplitudes and phases due to nonlinearity [23].

Substituting Eq. (21) in Eqs. (19) and (20), one obtains different order equations of ε.

Order ε:

$$\begin{aligned} &-D_0^2 w_{11} - a w_{11}'''' - x w_{11}' + (a\alpha_3\eta_1) w_{11}'' + \frac{1}{2}(1-x^2) w_{11}'' = 0, \quad 0 \leqslant x \leqslant \eta_1, \\ &-D_0^2 w_{21} - a w_{21}'''' - x w_{21}' + \frac{1}{2}(1-x^2) w_{21}'' = 0, \quad \eta_1 \leqslant x \leqslant \eta_2, \\ &-D_0^2 w_{31} - a w_{31}'''' - x w_{31}' + \frac{1}{2}(1-x^2) w_{31}'' = 0, \quad \eta_2 \leqslant x \leqslant 1, \end{aligned} \tag{22}$$

$$\begin{aligned} &w_{11}(0,t) = 0, \quad w_{11}'(0,t) = 0, \quad w_{31}''(1,t) = 0, \quad w_{31}'''(1,t) = 0, \\ &w_{11}(\eta_1,t) = w_{21}(\eta_1,t), \quad w_{11}'(\eta_1,t) = w_{21}'(\eta_1,t), \\ &w_{11}''(\eta_1,t) = w_{21}''(\eta_1,t), \quad w_{21}(\eta_2,t) = w_{31}(\eta_2,t), \\ &w_{21}'(\eta_2,t) = w_{31}'(\eta_2,t), \end{aligned}$$

$$w_{11}''' - w_{21}''' - (\alpha_3\eta_1) w_{11}' + \alpha_1 \Delta_{11} + \alpha_3 D_0^2 \Delta_{11} = 0 \quad \text{at } x = \eta_1,$$

$$w_{31}'' - w_{21}'' + (\eta_2 - \eta_1)\alpha_1 \Delta_{11} + (\eta_2 - \eta_1)\alpha_3 D_0^2 \Delta_{11} = 0 \quad \text{at } x = \eta_2,$$

$$w_{21}''' - w_{31}''' - \alpha_1 \Delta_{11} - \alpha_3 D_0^2 \Delta_{11} = 0 \quad \text{at } \ x = \eta_2, \tag{23}$$

where, $\Delta_{11} = w_{21}(\eta_2) - (\eta_2 - \eta_1) w_{21}'(\eta_2) - w_{11}(\eta_1)$ and $D_0^2 \Delta_{11} = D_0^2 w_{21}(\eta_2) - (\eta_2 - \eta_1) w_{21}'(\eta_2) - D_0^2 w_{11}(\eta_1)$.

Order ε^3:

$$D_0^2 w_{13} + a w_{13}'''' + x w_{13}' - \frac{1}{2}(1-x^2) w_{13}'' - (a\alpha_3\eta_1) w_{13}'' = -2D_0 D_2 w_{11}, \tag{24a}$$

$$D_0^2 w_{23} + a w_{23}'''' + x w_{23}' - \tfrac{1}{2}(1-x^2) w_{23}'' = -2D_0 D_2 w_{21}, \tag{24b}$$

$$D_0^2 w_{33} + a w_{33}'''' + x w_{33}' - \tfrac{1}{2}(1-x^2) w_{33}'' = -2D_0 D_2 w_{31}, \tag{24c}$$

$$\begin{aligned} &w_{13}(0,t) = 0, \quad w_{13}'(0,t) = 0, \quad w_{33}''(1,t) = 0, \quad w_{33}'''(1,t) = 0, \\ &w_{13}(\eta_1,t) = w_{23}(\eta_1,t), \quad w_{13}'(\eta_1,t) = w_{23}'(\eta_1,t), \\ &w_{13}''(\eta_1,t) = w_{23}''(\eta_1,t), \quad w_{23}(\eta_2,t) = w_{33}(\eta_2,t), \\ &w_{23}'(\eta_2,t) = w_{33}'(\eta_2,t), \end{aligned}$$

$$\begin{aligned} &w_{13}'''(\eta_1,t) - w_{23}'''(\eta_1,t) - \alpha_3\eta_1 w_{13}'(\eta_1,t) + \alpha_1 \Delta_{13}(\eta_1,t) + \alpha_2 \Delta_{11}^3(\eta_1,t) \\ &\quad + \alpha_3 D_0^2 \Delta_{13}(\eta_1,t) + 2\alpha_3 D_2 D_0 \Delta_{11}(\eta_1,t) = 0, \end{aligned}$$

$$w''_{33}(\eta_2,t) - w''_{23}(\eta_2,t) + (\eta_2-\eta_1)\alpha_1\Delta_{13}(\eta_2,t) + (\eta_2-\eta_1)\alpha_2\Delta_{11}^3(\eta_2,t)$$
$$+ (\eta_2-\eta_1)\alpha_3\{D_0^2\Delta_{13}(\eta_2,t) + 2D_2D_0\Delta_{11}(\eta_2,t)\} = 0,$$

$$w'''_{23}(\eta_2,t) - w'''_{33}(\eta_2,t) - \alpha_1\Delta_{13}(\eta_2,t) - \alpha_2\Delta_{11}^3(\eta_2,t)$$
$$- \alpha_3\{D_0^2\Delta_{13}(\eta_2,t) + 2D_2D_0\Delta_{11}(\eta_2,t)\} = 0, \quad (25)$$

where, $D_0 \equiv \partial/\partial T_0$ and $D_2 \equiv \partial/\partial T_2$.

2.2. Linear solution

At order ε, the equations and boundary conditions are linear and hence the solution is assumed of the form

$$w_{11} = \{A(T_2)e^{i\omega T_0} + c.c.\}y_1(x),$$
$$w_{21} = \{A(T_2)e^{i\omega T_0} + c.c.\}y_2(x),$$
$$w_{31} = \{A(T_2)e^{i\omega T_0} + c.c.\}y_3(x), \quad (26)$$

where $c.c.$ is the complex conjugate of the preceding terms and $y_i(x)$ are the displacement components only.

Introducing Eq. (26) in Eqs. (24) and (25) one gets

$$ay_1'''' - \frac{1}{2}(1-x^2)y_1'' - (a\alpha_3\eta_1)y_1'' + xy_1' - \omega^2 y_1 = 0, \quad 0\leqslant x\leqslant \eta_1,$$
$$ay_2'''' - \frac{1}{2}(1-x^2)y_2'' + xy_2' - \omega^2 y_2 = 0, \quad \eta_1\leqslant x\leqslant \eta_2,$$
$$ay_3'''' - \frac{1}{2}(1-x^2)y_3'' + xy_3' - \omega^2 y_3 = 0, \quad \eta_2\leqslant x\leqslant 1, \quad (27)$$

$$y_1(0)=0,\quad y_1'(0)=0,\quad y_2''(1)=0,\quad y_2'''(1)=0,\quad y_1(\eta_1)=y_2(\eta_1),$$
$$y_1'(\eta_1)=y_2'(\eta_1),\quad y_1''(\eta_1)=y_2''(\eta_1),\quad y_2(\eta_2)=y_3(\eta_2),\quad y_2'(\eta_2)=y_3'(\eta_2),$$

$$y_1'''(\eta_1) - y_2'''(\eta_1) - (\alpha_3\eta_1)y_1'(\eta_1) - \alpha_1 y_2(\eta_2) - \alpha_1(\eta_2-\eta_1)y_2'(\eta_2)$$
$$- \alpha_1 w_1(\eta_1) - \alpha_3\omega^2 y_2(\eta_2) + \alpha_3(\eta_2-\eta_1)\omega^2 y_2'(\eta_2) + \alpha_3\omega^2 y_1(\eta_1) = 0,$$
$$y_3''(\eta_2) - y_2''(\eta_2) + \alpha_1(\eta_2-\eta_1)y_2(\eta_2) - \alpha_1(\eta_2-\eta_1)^2 y_2'(\eta_2) - \alpha_1(\eta_2-\eta_1)y_1(\eta_1)$$
$$- \alpha_3(\eta_2-\eta_1)\omega^2 y_2(\eta_2) + \alpha_3(\eta_2-\eta_1)^2\omega^2 y_2'(\eta_2) + \alpha_3(\eta_2-\eta_1)\omega^2 y_1(\eta_1) = 0,$$
$$y_2''(\eta_2) - y_3''(\eta_2) - \alpha_1 y_2(\eta_2) + \alpha_1(\eta_2-\eta_1)y_2'(\eta_2) + \alpha_1 y_1(\eta_1) + \alpha_3\omega^2 y_2(\eta_2)$$
$$- \alpha_3(\eta_2-\eta_1)\omega^2 y_2'(\eta_2) - \alpha_3\omega^2 y_1(\eta_1) = 0. \quad (28)$$

The power series solutions of the Eqs. (27) and (28) can be expressed as

$$y_1(x) = \sum_{k=1}^{\infty} A_k x^{k-1}, \quad 0\leqslant x\leqslant \eta_1,$$
$$y_2(x) = \sum_{k=1}^{\infty} B_k x^{k-1}, \quad \eta_1\leqslant x\leqslant \eta_2,$$
$$y_3(x) = \sum_{k=1}^{\infty} C_k x^{k-1}, \quad \eta_2\leqslant x\leqslant 1. \quad (29)$$

Substituting Eq. (29) in Eqs. (27) and (28), one obtains the following recurrence relationship and the boundary conditions.

$$\begin{aligned}
&1.A_{k+4}-\frac{(0.5+a\alpha_3\eta_1)}{a(k+2)(k+3)}\,A_{k+2}+\frac{0.5[k(k-1)-2\omega^2]}{a.k(k+1)(k+2)(k+3)}\,A_k=0,\\
&1.B_{k+4}-\frac{0.5}{a(k+2)(k+3)}\,B_{k+2}+\frac{0.5[k(k-1)-2\omega^2]}{a.k(k+1)(k+2)(k+3)}\,B_k=0,\\
&1.C_{k+4}-\frac{0.5}{a(k+2)(k+3)}\,C_{k+2}+\frac{0.5[k(k-1)-2\omega^2]}{a.k(k+1)(k+2)(k+3)}\,C_k=0,
\end{aligned} \tag{30}$$

$A_1=0,\quad A_2=0,$

$$\begin{aligned}
&\sum_{k=1}^{\infty}C_k(k-1)(k-2)=0,\\
&\sum_{k=1}^{\infty}C_k(k-1)(k-2)(k-3)=0,\\
&\sum_{k=1}^{\infty}A_k\eta_1^{k-1}-\sum_{k=1}^{\infty}B_k\eta_1^{k-1}=0,\\
&\sum_{k=1}^{\infty}A_k(k-1)\eta_1^{k-2}-\sum_{k=1}^{\infty}B_k(k-1)\eta_1^{k-2}=0,\\
&\sum_{k=1}^{\infty}A_k(k-1)(k-2)\eta_1^{k-3}-\sum_{k=1}^{\infty}B_k(k-1)(k-2)\eta_1^{k-3}=0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty}A_k(k-1)(k-2)(k-3)\eta_1^{k-4}-\sum_{k=1}^{\infty}B_k(k-1)(k-2)(k-3)\eta_1^{k-4}\\
&\quad-(\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty}A_k\eta_1^{k-1}-(a\alpha_3)\sum_{k=1}^{\infty}A_k(k-1)\eta_1^{k-2}\\
&\quad+(\alpha_1-\alpha_3\omega^2)\sum_{k=1}^{\infty}B_k(k-1)\eta_2^{k-2}-(\eta_2-\eta_1)(\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty}B_k(k-1)\eta_2^{k-2}=0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty}B_k\eta_2^{k-1}-\sum_{k=1}^{\infty}C_k\eta_2^{k-1}=0,\\
&\sum_{k=1}^{\infty}B_k(k-1)\eta_2^{k-2}-\sum_{k=1}^{\infty}C_k(k-1)\eta_2^{k-2}=0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty}B_k(k-1)(k-2)\eta_2^{k-3}-\sum_{k=1}^{\infty}C_k(k-1)(k-2)\eta_2^{k-3}+(\eta_2-\eta_1)^2(\alpha_1-\omega^2\alpha_3)\\
&\sum_{k=1}^{\infty}B_k(k-1)\eta_2^{k-2}+(\eta_2-\eta_1)(\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty}A_k\eta_1^{k-1}-(\eta_2-\eta_1)(\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty}B_k\eta_2^{k-1}=0,
\end{aligned}$$

$$\sum_{k=1}^{\infty} B_k(k-1)(k-2)(k-3)\eta_2^{k-4} - \sum_{k=1}^{\infty} C_k(k-1)(k-2)(k-3)\eta_2^{k-4}$$
$$+ (\eta_2-\eta_1)(\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty} B_k(k-1)\eta_2^{k-2} + (\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty} A_k\eta_1^{k-1} - (\alpha_1-\omega^2\alpha_3)\sum_{k=1}^{\infty} B_k\eta_2^{k-1} = 0. \quad (31)$$

If the power series is truncated at the Pth term, then there are altogether $3P$ unknown coefficients. From the recurrence relations (30) and the boundary conditions (31) one obtains $3P$ simultaneous linear homogeneous equations. For a non-trivial solution, the determinant of the coefficient matrix must vanish. Thus, setting this diterminant equal to zero, one gets the frequency equation which is solved numerically for unknown linear frequency ω.

2.3. Nonlinear solution

Since the homogeneous part of Eqs. (24) has a non-trivial solution, the inhomogeneous Eqs. (24) have a solution only if a solvability condition is satisfied [21]. In order to find this condition, their solution is expressed in the form:

$$\begin{aligned} w_{13}(x,t) &= [A(T_2)e^{i\omega T_0} + c.c]\Phi_1(x) + \bar{W}_1(x,T_0,T_2),\\ w_{23}(x,t) &= [A(T_2)e^{i\omega T_0} + c.c]\Phi_2(x) + \bar{W}_2(x,T_0,T_2),\\ w_{33}(x,t) &= [A(T_2)e^{i\omega T_0} + c.c]\Phi_3(x) + \bar{W}_3(x,T_0,T_2). \end{aligned} \quad (32)$$

Substituting Eqs. (32) in Eqs. (17) and (18) and collecting the co-efficient of $e^{i\omega T_0}$ and equating to zero (i.e. removing the secular term), one can obtain

$$\begin{aligned} &\left\{a\Phi_1'''' + x\Phi_1' - \frac{1}{2}(1-x^2)\Phi_1'' - \omega^2\Phi_1 - (a\alpha_3\eta_1)\Phi_1'\right\} = -2(i\omega)\left(\frac{A'}{A}\right)y_1(x),\\ &\left\{a\Phi_2'''' + x\Phi_2' - \frac{1}{2}(1-x^2)\Phi_2'' - \omega^2\Phi_2\right\} = -2(i\omega)\left(\frac{A'}{A}\right)y_2(x),\\ &\left\{a\Phi_3'''' + x\Phi_3' - \frac{1}{2}(1-x^2)\Phi_3'' - \omega^2\Phi_3\right\} = -2(i\omega)\left(\frac{A'}{A}\right)y_3(x), \end{aligned} \quad (33)$$

$$\begin{aligned} &\Phi_1(0)=0, \quad \Phi_1'(0)=0, \quad \Phi_3''(1)=0, \quad \Phi_3'''(1)=0,\\ &\Phi_1(\eta_1)=\Phi_2(\eta_1), \quad \Phi_1'(\eta_1)=\Phi_2'(\eta_1), \quad \Phi_1''(\eta_1)=\Phi_2''(\eta_1)\\ &\Phi_2(\eta_2)=\Phi_3(\eta_2), \quad \Phi_2'(\eta_2)=\Phi_3'(\eta_2), \end{aligned}$$

$$\Phi_3'''(\eta_1) - \Phi_2'''(\eta_1) - (\alpha_3\eta_1)\Phi_1'(\eta_1) + \alpha_1\delta_{13} + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} + \alpha_3\left(\frac{A'}{A}\right)(2i\omega)\delta_{11} = 0,$$

$$\Phi_3''(\eta_2) - \Phi_2''(\eta_2) + (\eta_2-\eta_1)\left[\alpha_1\delta_{13} + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} + \alpha_3\left(\frac{A'}{A}\right)(2i\omega)\delta_{11}\right] = 0,$$

$$\Phi_3'''(\eta_2) - \Phi_2'''(\eta_2) - \left[\alpha_1\delta_{13} + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} + \alpha_3\left(\frac{A'}{A}\right)(2i\omega)\delta_{11}\right] = 0, \quad (34)$$

where

$$A' \equiv \frac{\partial A}{\partial T_2}, \quad A(T_2) = \frac{1}{2}\sigma(T_2)e^{i\theta(T_2)} \quad \text{and } A\bar{A} = \frac{1}{4}\sigma^2.$$

Defining

$$\delta_{11} = y_2(\eta_2) - (\eta_2-\eta_1)y_2'(\eta_2) - y_1(\eta_1),$$

$$\delta_{13} = \Phi_2(\eta_2) - (\eta_2 - \eta_1)\Phi_2'(\eta_2) - \Phi_1(\eta_1),$$

$$\begin{aligned}\Delta_{11}^3 &= [w_{21}(\eta_2, t) - (\eta_2 - \eta_1)w_{21}'(\eta_2, t) - w_{11}(\eta_1, t)]^3 \\ &= [A\delta_{11}e^{i\omega T_0} + c.c.]^3 + \cdots \\ &= [3A^2\bar{A}\delta_{11}^3 e^{i\omega T_0} + c.c.] + \cdots,\end{aligned}$$

$$\begin{aligned}\Delta_{13}^3 &= [w_{23}(\eta_2, t) - (\eta_2 - \eta_1)w_{23}'(\eta_2, t) - w_{13}(\eta_1, t)] \\ &= [A\delta_{13}e^{i\omega T_0} + c.c.] + \cdots,\end{aligned}$$

$$\begin{aligned}D_0\Delta_{13} &= -\omega^2 A\delta_{13}e^{i\omega T_0} + c.c., \\ D_2D_0\Delta_{11} &= A'(i\omega)\delta_{11}e^{i\omega T_0} + c.c.\end{aligned}$$

and using the relationship

$$-2(i\omega)\frac{\partial}{\partial T_2}(\ln A) = \frac{-2i\omega}{\sigma(T_2)}\left(\frac{\partial \sigma}{\partial T_2}\right) + 2\omega\frac{\partial \theta}{\partial T_2}$$

into Eqs. (33) and (34) and equating real and imaginary parts one gets,

$$\left\{a\Phi_1'''' + x\Phi_1' - \frac{1}{2}(1 - x^2)\Phi_1'' - \omega^2\Phi_1 - (a\alpha_3\eta_1)\Phi_1''\right\} = 2\omega\frac{\partial \theta}{\partial T_2}y_1(x), \tag{35a}$$

$$\left\{a\Phi_2'''' + x\Phi_2' - \frac{1}{2}(1 - x^2)\Phi_2'' - \omega^2\Phi_2\right\} = 2\omega\frac{\partial \theta}{\partial T_2}y_2(x), \tag{35b}$$

$$\left\{a\Phi_3'''' + x\Phi_3' - \frac{1}{2}(1 - x^2)\Phi_3'' - \omega^2\Phi_3\right\} = 2\omega\frac{\partial \theta}{\partial T_2}y_3(x) \tag{35c}$$

and $\partial\sigma/\partial T_2 = 0$.

Boundary conditions are

$$\begin{aligned}&\Phi_1(0) = 0, \quad \Phi_1'(0) = 0, \quad \Phi_3''(1) = 0, \quad \Phi_3'''(1) = 0, \quad \Phi_1(\eta_1) = \Phi_2(\eta_1), \\ &\Phi_1'(\eta_1) = \Phi_1'(\eta_1), \quad \Phi_1''(\eta_1) = \Phi_2''(\eta_1), \quad \Phi_2(\eta_2) = \Phi_3(\eta_2),\end{aligned}$$

$$\begin{aligned}\Phi_2'(\eta_2) = \Phi_3'(\eta_2)\Phi_1'''(\eta_1) - \Phi_2'''(\eta_1) - (\alpha_3\eta_1)\Phi_1'(\eta_1) + \alpha_1\delta_{13} \\ + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} - 2\omega\alpha_3\delta_{11}\frac{\partial\theta}{\partial T_2} = 0,\end{aligned}$$

$$\begin{aligned}&\Phi_3''(\eta_2) - \Phi_2''(\eta_2) + (\eta_2 - \eta_1)\left[\alpha_1\delta_{13} + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} + 2\omega\alpha_3\delta_{11}\frac{\partial\theta}{\partial T_2}\right] = 0, \\ &\Phi_3''(\eta_2) - \Phi_2'''(\eta_2) - \left[\alpha_1\delta_{13} + \alpha_2\left(\frac{3\sigma^2}{4}\right)\delta_{11}^3 - \alpha_3\omega^2\delta_{13} - 2\omega\alpha_3\delta_{11}\frac{\partial\theta}{\partial T_2}\right] = 0.\end{aligned} \tag{36}$$

Multiplying Eq. (35a) by y_1, Eq. (35b) by y_2, and Eq. (35c) by y_3 and integrating over $\int_0^{\eta_1} \mathrm{d}x$, $\int_{\eta_1}^{\eta_2} \mathrm{d}x$ and $\int_{\eta_2}^{1} \mathrm{d}x$, respectively, and subsequently adding, one obtains

$$\begin{aligned}
&\int_0^{\eta_1} \left\{ a\Phi_1'''' + x\Phi_1' - \frac{1}{2}(1-x^2)\Phi_1'' - \omega^2\Phi_1 - (a\alpha_3\eta_1)\Phi_1'' \right\} y_1 \,\mathrm{d}x \\
&\quad + \int_{\eta_1}^{\eta_2} \left\{ a\Phi_2'''' + x\Phi_2' - \frac{1}{2}(1-x^2)\Phi_2'' - \omega^2\Phi_2 \right\} y_2 \,\mathrm{d}x \\
&\quad + \int_{\eta_1}^{1} \left\{ a\Phi_3'''' + x\Phi_3' - \frac{1}{2}(1-x^2)\Phi_3'' - \omega^2\Phi_3 \right\} y_3 \,\mathrm{d}x \\
&= 2\omega\theta' \left(\int_0^{\eta_1} y_1^2 \,\mathrm{d}x + \int_{\eta_1}^{\eta_2} y_2^2 \,\mathrm{d}x + \int_{\eta_2}^{\eta_1} y_3^2 \,\mathrm{d}x \right). \qquad (37)
\end{aligned}$$

After some algebraic manipulation left-hand side of Eq. (37) becomes

$$\tfrac{3}{4} a\alpha_2\sigma^2\delta_{11}^4 - 2\omega a\alpha_3\theta'\delta_{11}^2.$$

The detail of the algebraic manipulation is given in Appendix A.

Therefore, one may write

$$\frac{3}{4}\, a\alpha_2\sigma^2\delta_{11}^4 - 2\omega a\alpha_3\theta'\delta_{11}^2 = 2\omega\theta' \left(\int_0^{\eta_1} y_1^2 \,\mathrm{d}x + \int_{\eta_1}^{\eta_2} y_2^2 \,\mathrm{d}x + \int_{\eta_2}^{1} y_3^2 \,\mathrm{d}x \right). \qquad (38)$$

Let us assume

$$b_1 = \int_0^{\eta_1} y_1^2 \,\mathrm{d}x + \int_{\eta_1}^{\eta_2} y_2^2 \,\mathrm{d}x + \int_{\eta_2}^{1} y_3^2 \,\mathrm{d}x.$$

Then Eq. (38) becomes

$$\theta' = \left(\frac{3a\alpha_2}{8\omega}\, \sigma^2 \right) \delta_{11}^4 \, \frac{1}{b_1 + a\alpha_3\delta_{11}^2}. \qquad (39)$$

On integration, one obtains

$$\theta = \left(\frac{3a\alpha_2}{8\omega}\, \sigma^2 \right) \delta_{11}^4 \, \frac{T_2}{b_1 + a\alpha_3\delta_{11}^2}.$$

Considering, $T_2 = \varepsilon^2 t$, $\sigma = \hat{A}/\varepsilon$ and $\alpha_4 = a\alpha_3$, the above equation becomes

$$\theta = \left(\frac{3a\alpha_2}{8\omega} \right) \delta_{11}^4 \, \frac{1}{b_1 + \alpha_4\delta_{11}^2} \, \hat{A}^2 t. \qquad (40)$$

Thus, from Eq. (40), one can obtain the frequency–amplitude relationship as

$$\omega_{nl} = \omega + \left(\frac{3a\alpha_2}{8\omega} \right) \delta_{11}^4 \, \frac{1}{b_1 + \alpha_4\delta_{11}^2} \, \hat{A}^2. \qquad (41)$$

3. Results and discussion

The numerical results obtained by using the methods outlined in the previous section are presented below in two parts. The data used are given in Table 1.

Table 1
Data for the beam to obtain the numerical results

$m = 9.7\,\text{kg/m}$, $L = 6.6\,\text{m}$, $a = 0.0106$, $\Omega = 32.8\,\text{rad/s}$, $\alpha_1 = 1000$

Table 2
The first three linear natural frequencies of the rotating beam without spring ($\alpha_1 = 0.0$) with comparison of the other references

	First mode	Second mode	Third mode
Present study	1.1244	3.4073	7.6170
Pohit et al. [17]	1.1245	3.4073	7.6218
Friedmann et al. [24]	1.1250	—	—
Gupta et al. [25]	1.1247	3.4089	7.6376

Table 3
The first three linear natural frequencies of the rotating beam with massless linear spring ($M = 0$) for various locations (η_2) of the torque tube; $\eta_1 = 0.1$

η_2	First mode	Second mode	Third mode
0.15	1.12465	3.40768	7.61706
0.20	1.12658	3.40845	7.62710
0.25	1.12972	3.40732	7.71556

Table 4
The first three linear natural frequencies of the rotating beam with spring with mass system for various locations (η_2) of the torque tube with $\alpha_4 = 0.15$

η_2	First mode	Second mode	Third mode
0.15	1.12494	3.40874	7.61962
0.20	1.12685	3.40943	7.62175
0.25	1.12998	3.40844	7.64637

3.1. Linear analysis

The natural frequencies of the rotating beam are obtained by using the power series method. First the analysis is carried out without spring. The roots of the frequency equations (Eqs. (27) and (28)) are obtained by the power series method following an iterative search procedure. The results are presented in Table 2. It is observed that these results so obtained are in excellent agreement, with those presented in Refs. [17,24,25]. In the present problem, the order of the determinant is taken as 300×300 to guarentee the convergence for the numerical calculation (convergence occured in the order of 240×240). For calculating the value of the natural frequency (ω), we have taken the tolerance limit for error as 10^{-7}.

Next the first three natural frequencies of the rotating beam with spring attached at different locations ($\eta_2 = 0.15$, 0.20 and 0.25) are determined. In this case, the mass of the spring is neglected. The value of dimensionless spring constant α_1 is assumed to be 1000. The same analysis is also carried out taking the mass of the spring into account. It is to be noted that in order to compare the mass of the spring with respect to that of the beam, a new non-dimensional number α_4 ($= a\alpha_3 = M/mL$) has been introduced in Eq. (40). The results are presented in Tables 3 and 4 corresponding to $\alpha_4 = 0.0$ (massless spring) and $\alpha_4 = 0.15$ (mass of the spring is 15% to that of mass of the beam) respectively. It is observed that due to the presence of the mass, the first natural frequencies for different spring locations ($\eta_2 = 0.15$, 0.20 and 0.25) have increased whereas those of the higher modes have diminished. It is evident that influence of mass of the spring on natural frequency is not very pronounced when the spring is located near the root (up to $\eta_2 = 0.15$). However, when the spring is located at $\eta_2 = 0.20$ or above, the mass of the spring should not be neglected while determining the linear frequencies as these values deviate from that of massless spring. Comparison of Tables 3 and 4 reveals that with the inclusion of spring mass, the magnitude of third natural frequency decreases while the first two frequencies exhibit reverse trend.

3.2. Nonlinear analysis

It has been observed that, elastomeric material exhibits nonlinear characteristics so far as the amplitude of motion is concerned. As the amplitude increases, degree of nonlinearity becomes more predominant. In this section, the aspect of nonlinear frequency with respect to amplitude of motion is addressed.

Table 5
The values of b_1 and $y_2(\eta_2)$ for the first three modes and for various locations of the torque tube on the beam with massless spring system

η_2		First mode	Second mode	Third mode
0.15	b_1	$0.52186e^{-004}$	$0.156716e^{-005}$	$0.16385e^{-007}$
	$y_2(\eta_2)$	$8.54820e^{-004}$	$-4.97205e^{-004}$	$-1.08344e^{-004}$
0.20	b_1	$0.45096e^{-004}$	$0.149762e^{-005}$	$0.19248e^{-007}$
	$y_2(\eta_2)$	-0.00130	$7.57158e^{-004}$	$1.63049e^{-004}$
0.25	b_1	$0.37072e^{-004}$	$0.15993e^{-005}$	$0.33031e^{-007}$
	$y_2(\eta_2)$	0.00172	0.00105	$2.45745e^{-004}$

Table 6
The values of b_1 and $y_2(\eta_2)$ for the first three modes and for various locations of the torque tube on the beam with spring mass system ($\alpha_4 = 0.15$)

η_2		First mode	Second mode	Third mode
0.15	b_1	$0.50197e^{-004}$	$0.15138e^{-005}$	$0.15794e^{-007}$
	$y_2(\eta_2)$	$-8.36419e^{-004}$	$4.87704e^{-004}$	$1.06267e^{-004}$
0.20	b_1	$0.43622e^{-004}$	$0.14552e^{-005}$	$0.16323e^{-007}$
	$y_2(\eta_2)$	-0.00127	$7.44880e^{-004}$	$1.51596e^{-004}$
0.25	b_1	$0.36045e^{-004}$	$0.15355e^{-005}$	$0.19655e^{-007}$
	$y_2(\eta_2)$	0.00169	-0.00103	$1.9631e^{-004}$

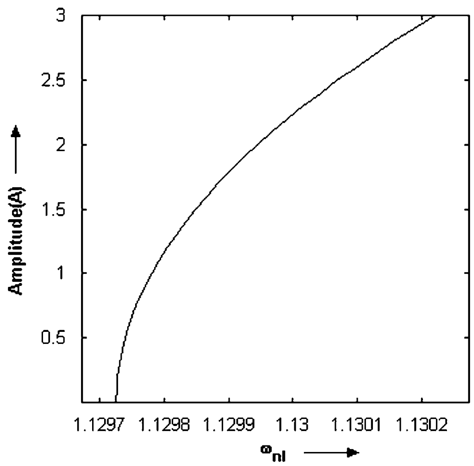

Fig. 2. Variation of first nonlinear frequency with amplitude of oscillation (massless spring $\alpha_4 = 0.0$, $\eta_2 = 0.25$).

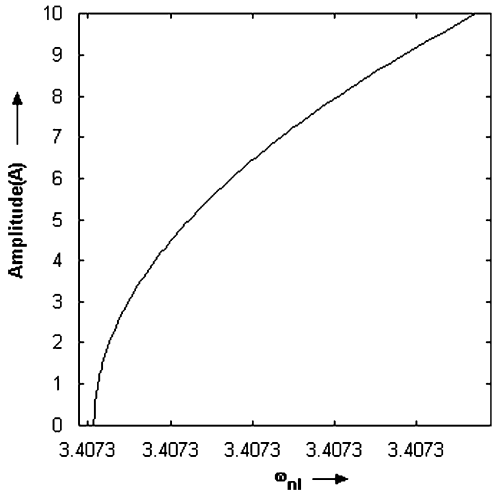

Fig. 3. Variation of second nonlinear frequency with amplitude of oscillation (massless spring $\alpha_4 = 0.0$, $\eta_2 = 0.25$).

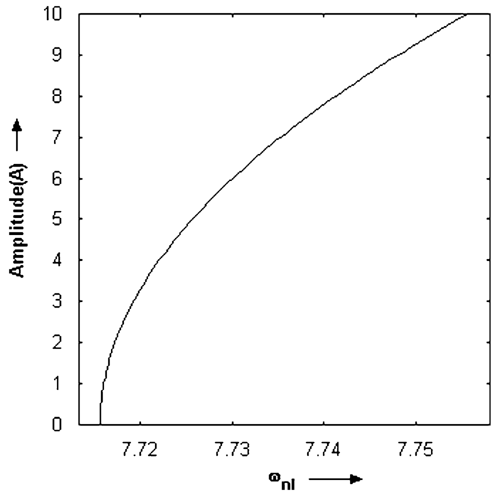

Fig. 4. Variation of third nonlinear frequency with amplitude of oscillation (massless spring $\alpha_4 = 0.0$, $\eta_2 = 0.25$).

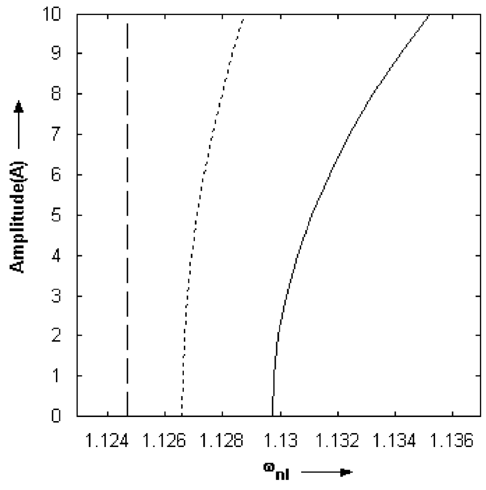

Fig. 5. Variation of first nonlinear frequency with amplitude of oscillation for different locations of the torque tube with massless spring ($\alpha_4 = 0.0$).

The nonlinear frequency–amplitude relationship of the rotating beam with spring–mass system is given in Eq. (41). Calculations are performed with a value of nonlinear spring constant (α_2) as 10^9. It may be noted that nonlinear spring constant of the elastomeric material actually used in helicopter rotor blade exhibits even higher value, Pohit et al. [17].

In order to obtain the nonlinear frequencies of the blade–spring–mass system, the value of b_1 and $y_2(\eta_2)$ are calculated for the first three modes of vibrations. They are presented in Table 5. In this Table, the mass of the spring is neglected, i.e. $\alpha_4 = 0.0$. The corresponding values of b_1 and $y_2(\eta_2)$ with spring–mass system ($\alpha_4 = 0.15$) are furnished in Table 6.

Having obtained the necessary numerical data (Tables 5 and 6), the nonlinear natural frequencies are obtained. The location of the torque tube is at $\eta_2 = 0.25$ and that of the spring is at $\eta_1 = 0.1$. Figs. 2–4 show

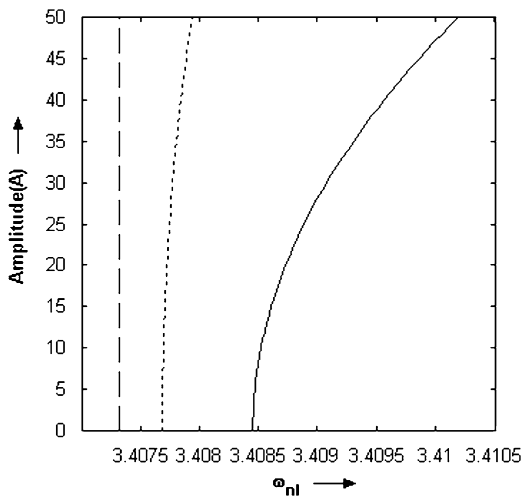

Fig. 6. Variation of second nonlinear frequency with amplitude of oscillation for different locations of the torque tube with massless spring ($\alpha_4 = 0.0$). —— $\eta_2 = 0.15$; ...... $\eta_2 = 0.20$; — $\eta_2 = 0.25$.

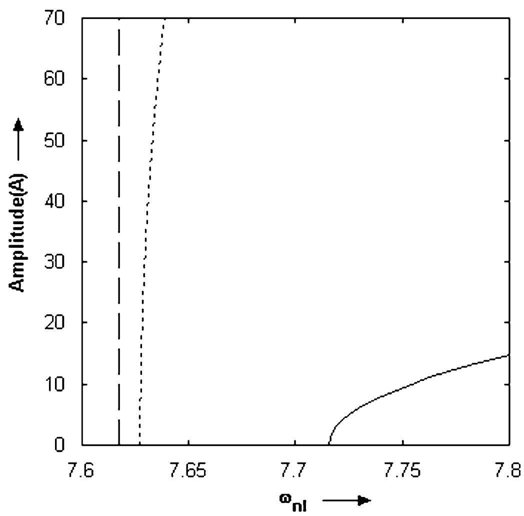

Fig. 7. Variation of third nonlinear frequency with amplitude of oscillation for different locations of the torque tube with massless spring ($\alpha_4 = 0.0$). —— $\eta_2 = 0.15$; ...... $\eta_2 = 0.20$; — $\eta_2 = 0.25$.

the variation of first three natural frequencies of the rotating beam with the tip amplitudes when mass of spring is neglected.

The dynamic characteristics of the rotating beam are analyzed for different torque tube locations. In Figs. 5–7, variations of nonlinear frequencies are plotted with the tip amplitude for different torque tube locations ($\eta_2 = 0.15$, 0.20 and 0.25) when spring location remains constant. It is observed that location of the torque tube has pronounced effect on frequencies and degree of nonlinearity increases with the increase in the value of η_2. Similar results are obtained considering the mass of the spring ($\alpha_4 = 0.15$). Figs. 8–10 exhibit the variation of the first three natural frequencies with tip amplitudes when torque tube is located at 0.25 ($\eta_2 = 0.25$).

Figs. 11–13 show the variation of the first three nonlinear frequencies with tip amplitudes for different values of torque tube locations ($\eta_2 = 0.15$, 0.20 and 0.25). However, the natures of the curves are found to follow the same pattern as those of Figs. 5–7.

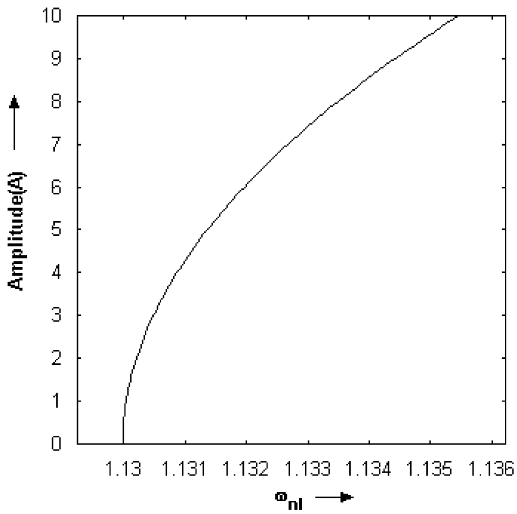

Fig. 8. Variation of first nonlinear frequency with amplitude of oscillation (spring with mass $\alpha_4 = 0.15$, $\eta_2 = 0.25$).

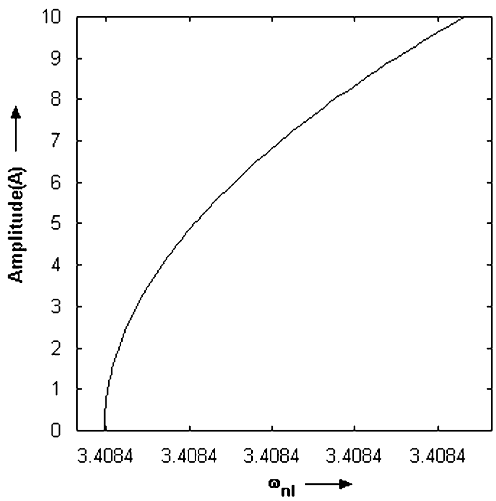

Fig. 9. Variation of second nonlinear frequency with amplitude of oscillation (spring with mass $\alpha_4 = 0.15$, $\eta_2 = 0.25$).

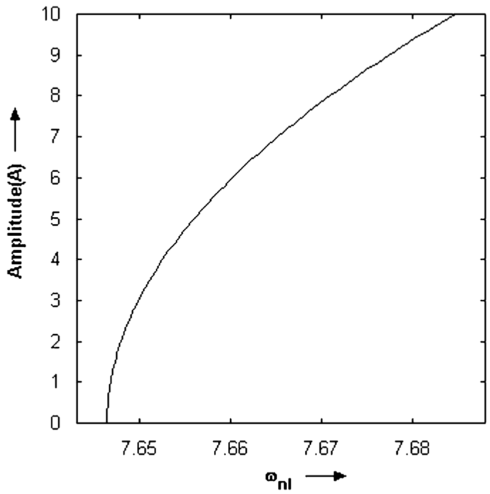

Fig. 10. Variation of third nonlinear frequency with amplitude of oscillation (spring with mass $\alpha_4 = 0.15$, $\eta_2 = 0.25$).

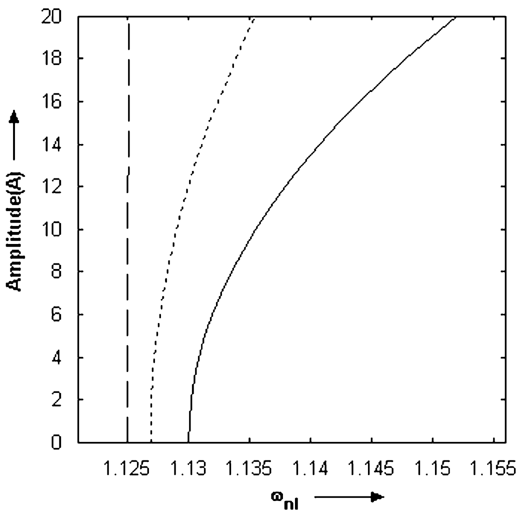

Fig. 11. Variation of first nonlinear frequency with amplitude of oscillation for different locations of the torque tube with spring mass ($\alpha_4 = 0.15$). —— $\eta_2 = 0.15$; …… $\eta_2 = 0.20$; — $\eta_2 = 0.25$.

A study has been also carried out to highlight the influence of spring mass on nonlinear frequency. The location of the torque tube is kept at $\eta_2 = 0.25$. Variations of nonlinear frequencies with tip amplitudes are plotted with nonlinear spring with mass ($\alpha_4 = 0.15$) and without mass ($\alpha_4 = 0.0$). The results of the first three modes of vibration are shown in Figs. 14–16. In all modes of vibration, as the amplitude of vibration increases, the effect of spring mass becomes more prominent. For the higher modes of vibration, the natures of the curves are very appreciably indicating the fact that it is not wise to neglect the mass of the spring in order to simplify the problem. It has been observed that the position of the backbone curve for the third mode has been shifted (Fig. 16) when compared to the first two frequencies.

It is interesting to know the effect of design changes on nonlinear frequencies of the rotating blade. In the present problem, the spring is attached with a rigid massless link, which is also rotating along with the beam AC. One end of the link is free and the other end being attached to the beam at a point towards the free end. In Ref. [22], the authors made a similar study in which the one end of a nonlinear spring is attached to the

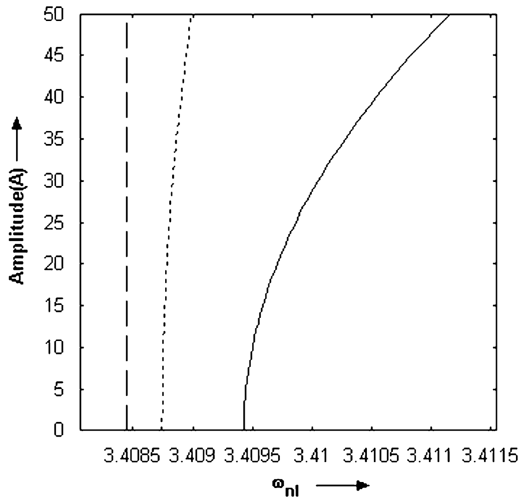

Fig. 12. Variation of second nonlinear frequency with amplitude of oscillation for different locations of the torque tube with spring mass ($\alpha_4 = 0.15$). —— $\eta_2 = 0.15$; …… $\eta_2 = 0.20$; — $\eta_2 = 0.25$.

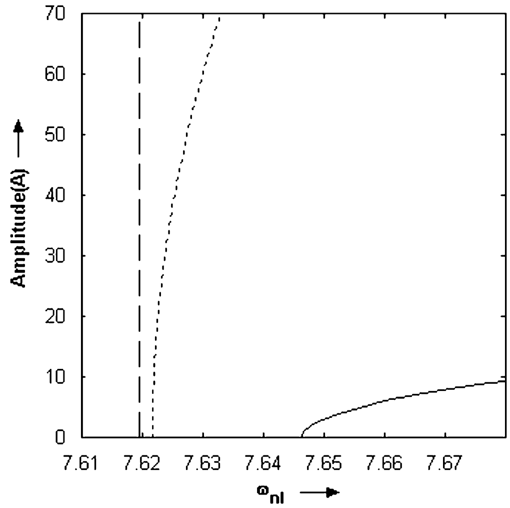

Fig. 13. Variation of third nonlinear frequency with amplitude of oscillation for different locations of the torque tube with spring mass ($\alpha_4 = 0.15$). —— $\eta_2 = 0.15$; …… $\eta_2 = 0.20$; — $\eta_2 = 0.25$.

rotating beam and the other end is attached to a rigid link that is also rotating with the beam. The first nonlinear frequencies for the present case ($\eta_2 = 0.20$ and 0.25; $\eta_1 = 0.1$) are superimposed on the frequency plot of Ref. [22] and are shown in Fig. 17. In Ref. [22], the location of the spring is taken as $\eta = 0.20$ and 0.25. It has been observed from the figure that not only the linear frequencies differ from one another but also the nature of the frequency plots vary. With the present configuration of the rigid link, degree of nonlinearity is decreased for both the locations of the torque tube.

4. Concluding remarks

Linear and nonlinear free vibration of a rotating beam under flap bending has been investigated. The beam is rotating with a transverse constraint in form of a nonlinear spring having finite mass. Formulation of the

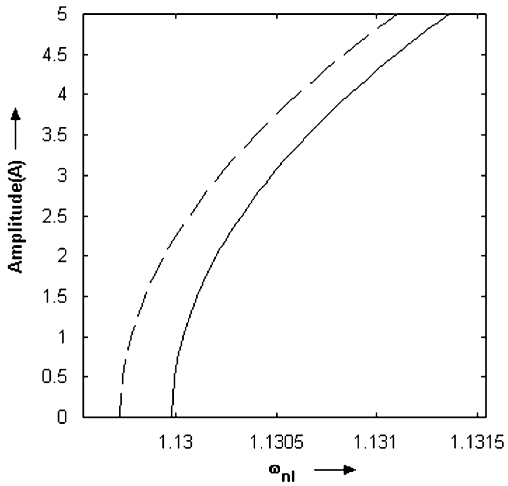

Fig. 14. Influence of first nonlinear frequency with $\eta_2 = 0.25$. —— $\alpha_4 = 0.0$; —— $\alpha_4 = 0.15$.

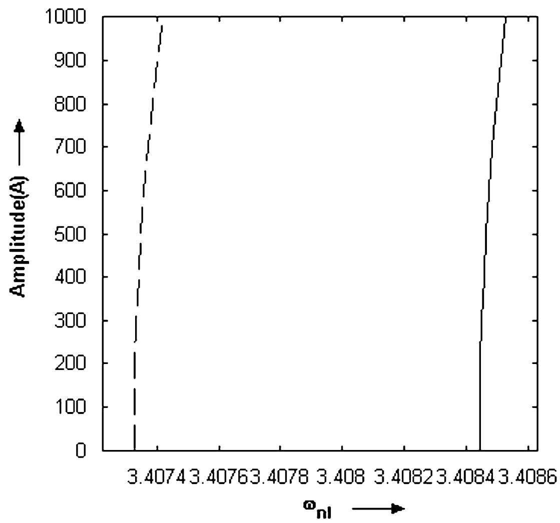

Fig. 15. Influence of second nonlinear frequency with $\eta_2 = 0.25$. —— $\alpha_4 = 0.0$; —— $\alpha_4 = 0.15$.

equation of motion of the rotating beam along with the nonlinear constraint have been presented in the paper starting from transverse/axial coupling through axial strain.

For linear problem, first natural frequencies are obtained for rotating beam without spring mass attachment. The results found are excellent agreement with the published results of other researchers. The first six linear frequencies are also calculated for a few location of the spring–mass system. It is revealed that the effect of spring mass is quite prominent on certain spring location, and it is not advisable to neglect spring mass while calculating natural frequencies of the system. The method of multiple time scale is directly applied to the partial differential equations and boundary conditions to determine the nonlinear frequencies of the system.

A closed form frequency–amplitude relationship of the rotating beam along with spring–mass system is obtained. The effect of the mass of the spring on nonlinear frequencies is also investigated. It may be concluded that mass of the spring plays a significant role in predicting frequency–amplitude relationship.

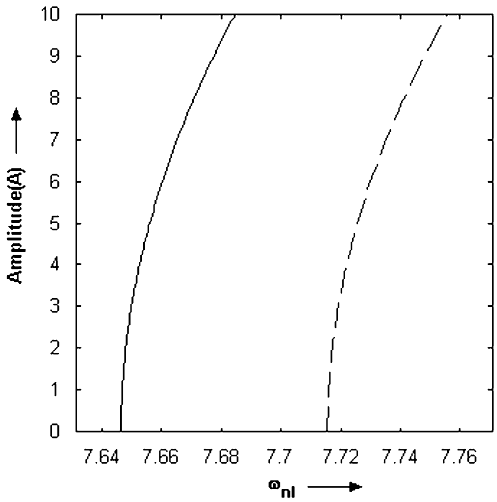

Fig. 16. Influence of third nonlinear frequency with $\eta_2 = 0.25$. —— $\alpha_4 = 0.0$; —— $\alpha_4 = 0.15$.

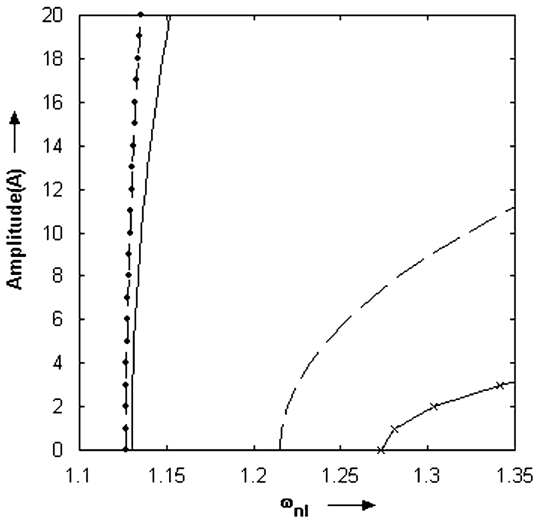

Fig. 17. Comparison of the frequency–amplitude relationships of the first mode of vibration under different design conditions. Present study: ●—● ●— ● $\eta_2 = 0.20$; —— $\eta_2 = 0.25$; ——$\eta_1 = 0.10$. Ref. [22]: – – – $\eta = 0.20$; — × — $\eta = 0.25$.

Further study is conducted to highlight the influence of the location of the spring–mass system on the rotating frequencies of the beam and it is noted that location of the spring exhibits significant effect on nonlinear frequencies.

An investigation is also carried out to assess the effect of design changes on nonlinear frequencies of the rotating blade. It is noted that with the present configuration of spring and torque tube, degree of nonlinearity is reduced.

Acknowledgements

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Appendix A

Left-hand side of Eq. (37)

$$\begin{aligned}
&\int_0^{\eta_1}\left\{a\Phi_1'''' + x\Phi_1' - \frac{1}{2}(1-x^2)\Phi_1'' - \omega^2\Phi_1 - (a\alpha_3\eta_1)\Phi_1''\right\}y_1\,\mathrm{d}x \\
&+ \int_{\eta_1}^{\eta_2}\left\{a\Phi_2'''' + x\Phi_2' - \frac{1}{2}(1-x^2)\Phi_2'' - \omega^2\Phi_2\right\}y_2\,\mathrm{d}x \\
&+ \int_{\eta_2}^{1}\left\{a\Phi_3'''' + x\Phi_3' - \frac{1}{2}(1-x^2)\Phi_3'' - \omega^2\Phi_3\right\}y_3\,\mathrm{d}x \\
&= a\int_0^{\eta_1}\Phi_1''''y_1\,\mathrm{d}x + a\int_{\eta_1}^{\eta_2}\Phi_2''''y_2\,\mathrm{d}x + a\int_{\eta_1}^{1}\Phi_3''''y_3\,\mathrm{d}x \\
&- \int_0^{\eta_1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_1'\right]y_1\,\mathrm{d}x - \int_{\eta_1}^{\eta_2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_2'\right]y_2\,\mathrm{d}x \\
&- \int_{\eta_2}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_3'\right]y_3\,\mathrm{d}x - (a\alpha_3\eta_1)\int_0^{\eta_1}\Phi_1''y_1\,\mathrm{d}x \\
&- \omega^2\int_0^{\eta_1}\Phi_1 y_1\,\mathrm{d}x - \omega^2\int_{\eta_1}^{\eta_2}\Phi_2 y_2\,\mathrm{d}x - \omega^2\int_{\eta_2}^{1}\Phi_3 y_3\,\mathrm{d}x.
\end{aligned}$$

After using boundary conditions on $y(x)$ and $\Phi(x)$ from Eqs. (28) and (36), with some algebraic manipulation one gets the following:

$$\begin{aligned}
&a\int_0^{\eta_1}\Phi_1''''y_1\,\mathrm{d}x + a\int_{\eta_1}^{\eta_2}\Phi_2''''y_2\,\mathrm{d}x + a\int_{\eta_2}^{1}\Phi_3''''y_3\,\mathrm{d}x = (a\alpha_3\eta_1)y_1(\eta_1)\Phi_1'(\eta_1) \\
&- (a\alpha_3\eta_1)\Phi_1(\eta_1)y_1'(\eta_1) - 2a\omega\alpha_3\delta_{11}^2\theta' + \left(\frac{3}{4}\sigma^2\right)a\alpha_2\delta_{11}^4 + a\int_0^{\eta_1}y_1''''\Phi_1\,\mathrm{d}x \\
&+ a\int_{\eta_1}^{\eta_2}y_2''''\Phi_2\,\mathrm{d}x + a\int_{\eta_2}^{1}y_3''''\Phi_3\,\mathrm{d}x,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&- \int_0^{\eta_1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_1'\right]y_1\,\mathrm{d}x - \int_{\eta_1}^{\eta_2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_2'\right]y_2\,\mathrm{d}x - \int_{\eta_2}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)\Phi_3'\right]y_3\,\mathrm{d}x \\
&= - \int_{\eta_2}^{\eta_1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)y_1'\right]\Phi_1\,\mathrm{d}x - \int_{\eta_1}^{\eta_2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)y_2'\right]\Phi_2\,\mathrm{d}x - \int_{\eta_2}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(1-x^2)y_3'\right]\Phi_3\,\mathrm{d}x
\end{aligned} \tag{A.2}$$

and

$$-(a\alpha_3\eta_1)\int_0^{\eta_1}\Phi_1''y_1\,\mathrm{d}x - (a\alpha_3\eta_1)y_1(\eta_1)\Phi_1'(\eta_1) + (a\alpha_3\eta_1)y_1'(\eta_1)\Phi_1(\eta_1) - (a\alpha_3\eta_1)\int_0^{\eta_1}y_1''\Phi_1\,\mathrm{d}x, \tag{A.3}$$

using Eqs. (A.1)–(A.3), left-hand side of Eq. (37) reduces to

$$\begin{aligned}
&\int_0^{\eta_1}\left[ay_1'''' + xy_1' - \frac{1}{2}(1-x^2)y_1'' - \omega^2 y_1 - (a\alpha_3\eta_1)y_1''\right]\Phi_1\,\mathrm{d}x \\
&+ \int_{\eta_1}^{\eta_2}\left[ay_2'''' + xy_2' - \frac{1}{2}(1-x^2)y_2'' - \omega^2 y_2\right]\Phi_2\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\eta_2}^{1}\left[ay_3'''' + xy_3' - \frac{1}{2}(1-x^2)y_3'' - \omega^2 y_3\right]\Phi_3\,\mathrm{d}x \\
&+\left(\frac{3}{4}\sigma^2\right)\alpha_2 a\delta_{11}^4 - 2a\omega\alpha_3\theta'\delta_{11}^2 \\
&=\left(\frac{3}{4}\sigma^2\right)\alpha_2 a\delta_{11}^4 - 2a\omega\alpha_3\theta'\delta_{11}^2 \quad \text{(simplified form obtained after using Eq. (27)).}
\end{aligned}
$$

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